Effect of Disturbances on the Heat Transfer of a Laminar Axisymmetric Boundary Layer

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The effect of a periodic crossflow velocity on the heat-transfer characteristics of a laminar axisymmetric boundary layer has been investigated. The three-dimensional axisymmetric boundary-layer equations were solved using a perturbation expansion technique to mathematically separate the standard boundary-layer flow from the imposed disturbance flow. The sets of equations were put into an almost two-dimensional form using a Mangler type transformation and were reduced to ordinary differential equations with appropriate asymptotic expansions. For thin boundary layers, the contribution of the disturbance was determined in closed form. The resulting expression for Nusselt number contains the standard flat plate term plus a disturbance term proportional to Reynolds number to the 3/2 power. The proposed model is used to correlate recently obtained experimental heat-transfer data for a cone-cylinder in supersonic and hypersonic flow. An order of magnitude estimate of the wavelength of the crossflow velocity perturbation indicates that this wavelength is inversely proportional to the unit Reynolds number of the external flow for a particular model geometry. Therefore the Reynolds number based on this wavelength is $Re_\lambda \sim 0(10^3)$.

Nomenclature

c_f = skin-friction coefficient
f = function of η
g = function of η
h = enthalpy
L = characteristic reference length
M = Mach number
Nu_x = Nusselt number based on x
Pr = Prandtl number
p_1 = initial shock tube pressure
r = radial coordinate
r_0 = cylinder radius
Re_x = Reynolds number based on x and freestream quantities
T = temperature
u = axial velocity
U = external axial velocity
v = normal velocity
V = external normal velocity
w = crossflow velocity
W = external crossflow velocity
x = axial coordinate
y = radial coordinate measured from cylinder wall
z = circumferential coordinate, $z = r_0 \phi$
$\bar{v}$ = transformed normal velocity defined as $\bar{v} = (Lr/r_0^2)v$
$\bar{w}$ = transformed crossflow velocity defined as $\bar{w} = (L/r)w$
$\bar{x}$ = transformed axial coordinate defined as $\bar{x} = (r_0^2/L^2)x$
$\bar{y}$ = transformed radial coordinate defined as $\bar{y} = \int (r/L)\,dy = (r_0 y/L) + (y^2/2L)$
$\bar{z}$ = transformed circumferential coordinate defined as $\bar{z} = r_0 \bar{\phi}$
γ = specific heat ratio
δ = boundary-layer thickness
ε = perturbation parameter, $\varepsilon = W_\infty/U_\infty$
η = transformed coordinate defined as $\eta = \bar{y}[U_\infty/v\bar{x}]^{1/2}$
θ = dimensionless temperature, $\theta = (T_0 - T_w)/(T_\infty - T_w)$
λ = disturbance wavelength
Λ = dimensionless temperature, $\Lambda = T_1/(T_\infty - T_w)$
μ = coefficient of viscosity
v = kinematic viscosity
ξ = transformed coordinate defined as $\xi = (2L/r_0^2)(v\bar{x}/U_\infty)^{1/2}$
ρ = mass density
ϕ = azimuthal angle
ψ = streamfunction
Ω = $\int_0^{\bar{y}} (\partial \bar{w}_1/\partial \bar{z})\,d\bar{y}$
$\bar{\phi}$ = transformed azimuthal angle defined as $\bar{\phi} = (r_0^2/L)(\phi/r_0)$

Subscripts
0 = zero-order term
1 = first-order term
t = stagnation condition
w = cylinder wall
∞ = freestream

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I. Introduction

STRIATION patterns, indicative of orderly three-dimensional disturbances, have been observed in a variety of two-dimensional supersonic laminar boundary-layer flows. The striation patterns have been observed qualitatively using either subliming or ablating processes to record the effect.

The disturbance patterns were first observed by Ginoux[1,2] in studies of supersonic laminar reattachment. Hopkins et al.[3] duplicated Ginoux's test conditions and obtained excellent photographs of the striation patterns using a fluorescent-oil visualization technique. Further experiments showed that striations were present in supersonic laminar boundary layers on a flat plate (with no steps or protuberances), behind a wire roughness, and in reattaching flows for both forward and backward facing steps. Miller et al.[4] reported striation patterns being "scorched" on flap models during hot-shot wind-tunnel tests. Further experiments by Ginoux[5] on reattaching boundary layers have shown that the static pressure, pitot pressure and heat-transfer rates vary periodically transverse to the attached flow. Evidence of orderly boundary-layer disturbances have also been observed in ablating flows by McDevitt and Mellenthin[6] for hypersonic conical flow and by Winkler et al.[7] for internal cylindrical flow.

One interpretation of the striation pattern has been given by Ginoux.[1,2,5] It was recognized that the striation patterns could be caused by the existance of Gortler type vortices[8] in the laminar boundary layer. Accordingly, the disturbances causing striation patterns were termed "streamwise vortices." However, the experimental evidence indicates only that a periodic variation of pitot pressure, heat and mass transfer exists at the surface. Three-dimensional velocity field measurements, which could be used to precisely determine the flow structure, have not been performed. The photographs and patterns observable on ablated models suggest that the striation patterns may be caused by streamwise vortices embedded in the boundary layer, however, it can be shown that these patterns can be explained just as well by the hypothesis of a periodically varying thickness of the laminar boundary layer due to spanwise periodic disturbances.

Using the hypothesis that "streamwise vortices" can exist in laminar boundary layers with concave curvature, Persen[9] proposed that the higher heat-transfer rates measured by Kestin et al.[10,11] were caused by these disturbances. Kestin et al. reported a strong dependance between heat-transfer and free-stream turbulence. It should be noted that the assumption of a streamwise vortex pattern is not necessary. Again, the same effects, i.e., higher heat-transfer rates and local areas of high stress, can be caused by the presence of a periodic transverse disturbance in the laminar boundary layer.

Striations have also been observed[12-15] to occur on re-entry cones upstream of crosshatch patterns. Tobak[16] has hypothesized that the striations are caused by "streamwise vortices" and that these disturbances are a prerequisite to the formation of crosshatch patterns. Coupled with the observation that cross-hatching appears only in turbulent supersonic boundary layers, it has been proposed[7,17] that the orderly disturbances appearing upstream of the crosshatch patterns are part of the transition process between an upstream laminar boundary layer and a downstream turbulent boundary layer.

Measurements of heat transfer[18] on the cylindrical surface downstream of the cone-cylinder corner in supersonic and hypersonic flows have resulted in considerably higher heat-transfer rates than would be expected for a laminar axisymmetric boundary layer. In view of the aforementioned experimental evidence, it is suspected that despite the laminar flow conditions, a disturbed laminar boundary layer exists. The source of the instability may be due to the concave streamline curvature existing just beyond the overexpansion at the cone-cylinder corner. Tobak[16] has indicated that very small amounts of concave curvature in the streamlines at the outer edge of the boundary layer can enhance the growth of disturbances.

In view of the experimental evidence concerning the susceptibility of the laminar boundary layer to orderly disturbances, a theoretical model is proposed which predicts heat-transfer and skin-friction characteristics of a laminar axisymmetric boundary layer perturbed in the transverse direction. A standard boundary-layer flow is considered with a disturbance superimposed on it in the form of a small perturbation.[19,20] The periodic variation in the transverse boundary-layer thickness is ensured by the upper boundary condition, $w(\infty) = W_\infty \sin 2\pi/\lambda z$ (cf. Sec. II), on the crossflow velocity. The effect of the changing boundary-layer thickness is to cause a periodic variation in heat transfer and skin friction at the surface.

The analysis is presented in Sec. II. In Sec. III a correlation of the theoretical results with experimental heat-transfer data is used to demonstrate the plausibility of the proposed model. An order of magnitude estimate of the disturbance wavenumber is shown in Table 1.

Table 1 Estimated disturbance wavelength and wave number

M_∞	Nu/Pr	h_t/h_w	Re/cm	λ, mm	$2\pi r_0/\lambda$
1.73	422	7.0	1.99×10^5	0.056	572
1.92	645	11.0	1.01×10^5	0.110	291
1.96	468	12.3	8.40×10^4	0.132	242
1.97	303	13.0	7.56×10^4	0.146	219
2.22	76.8	21.0	2.80×10^4	0.395	81
2.25	116	21.75	2.64×10^4	0.419	76
2.40	17.4	28.0	1.31×10^4	0.841	38
2.45	31.7	29.5	1.25×10^4	0.887	36
5.5	9.02	15.6	3.05×10^3	3.62	8
5.5	7.80	15.6	1.95×10^3	5.92	5

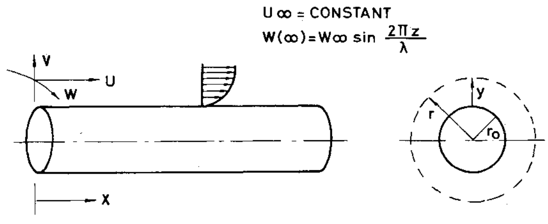

Fig. 1 Coordinate system.

II. Analysis

The following analysis considers the axisymmetric laminar flow about a circular cylinder (Fig. 1). The principal flow direction is along the length of the cylinder. In addition, the analysis will allow the existence of a transverse disturbance which appears in the form of a periodic crossflow velocity.

For the purposes of this analysis, an incompressible constant property flow is treated. The effects of compressibility and variable properties in the boundary layer will quantitatively influence the coefficients of Reynolds number appearing in the final result, however, the functional dependance of Nusselt number and skin-friction coefficient with Reynolds numbers is substantially unaffected. This has been justified in previous analysis by successful correlations of compressible boundary-layer data using incompressible relations,[21] which is the basis of the definitions of the proper reference conditions.

For incompressible flow with constant properties, the axisymmetric boundary layer equations may be written as

Continuity:

$$\partial u/\partial x + \partial v/\partial y + (1/r)\partial w/\partial\phi + v/r = 0 \tag{1}$$

Momentum (x-direction):

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{w}{r}\frac{\partial u}{\partial \phi}\right) = \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{1}{r}\frac{\partial u}{\partial y}\right) \tag{2}$$

Momentum (ϕ-direction):

$$\rho\left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + \frac{w}{r}\frac{\partial w}{\partial \phi}\right) = -\frac{1}{r}\frac{\partial p}{\partial \phi} + \mu\left(\frac{\partial^2 w}{\partial y^2} + \frac{1}{r}\frac{\partial w}{\partial y}\right) \tag{3}$$

Energy:

$$\rho\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \frac{w}{r}\frac{\partial T}{\partial \phi}\right) = \frac{\mu}{Pr}\left(\frac{\partial^2 T}{\partial y^2} + \frac{1}{r}\frac{\partial T}{\partial y}\right) \tag{4}$$

For the present, the transverse curvature terms have been retained. The viscous dissipation terms have been neglected and the pressure field in the axial direction assumed constant.

The boundary conditions are

$$u = v = w = 0, \quad T = T_w \quad \text{at } y = 0 \tag{5a}$$

$$u = U_\infty, \quad T = T_\infty, \quad w(\infty) = W_\infty \sin 2\pi z/\lambda \quad \text{as } y \to \infty \tag{5b}$$

where λ is the wavelength of the streamwise disturbances, and $z = r_0\phi$. It is now desired to treat the problem as one of an axisymmetric boundary-layer flow with an imposed transverse disturbance. Accordingly, a perturbation expansion of the following form is used:

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots \tag{6a}$$

$$v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \ldots \tag{6b}$$

$$w = \varepsilon w_1 + \varepsilon^2 w_2 + \ldots \tag{6c}$$

$$T = T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \ldots \tag{6d}$$

where

$$\varepsilon = W_\infty/U_\infty \ll 1 \tag{6e}$$

and the axial velocity, u, is assumed a function of x and y only.

Applying these expansions to the continuity, momentum and energy equations yields the following zero- and first-order sets of equations

Zero order:

$$\partial u_0/\partial x + \partial v_0/\partial y + v_0/r = 0 \tag{7}$$

$$u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial y} = v\left(\frac{\partial^2 u_0}{\partial y^2} + \frac{1}{r}\frac{\partial u_0}{\partial y}\right) \tag{8}$$

$$u_0 \frac{\partial T_0}{\partial x} + v_0 \frac{\partial T_0}{\partial y} = \frac{v}{Pr}\left(\frac{\partial^2 T_0}{\partial y^2} + \frac{1}{r}\frac{\partial T_0}{\partial y}\right) \tag{9}$$

$$u_0 = v_0 = 0, \quad T_0 = T_w \quad \text{at } y = 0 \tag{10a}$$

$$u_0 = U_\infty, \quad T_0 = T_\infty, \quad \text{as } y \to \infty \tag{10b}$$

First order:

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{1}{r}\frac{\partial w_1}{\partial \phi} + \frac{v_1}{r} = 0 \tag{11}$$

$$u_0 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_1}{\partial y} + v_1 \frac{\partial u_0}{\partial y} = v\left(\frac{\partial^2 u_1}{\partial y^2} + \frac{1}{r}\frac{\partial u_1}{\partial y}\right) \tag{12}$$

$$u_0 \frac{\partial w_1}{\partial x} + v_0 \frac{\partial w_1}{\partial y} = v\left(\frac{\partial^2 w_1}{\partial y^2} + \frac{1}{r}\frac{\partial w_1}{\partial y}\right) \tag{13}$$

$$u_0 \frac{\partial T_1}{\partial x} + u_1 \frac{\partial T_0}{\partial x} + v_0 \frac{\partial T_1}{\partial y} + v_1 \frac{\partial T_0}{\partial y} = \frac{v}{Pr}\left(\frac{\partial^2 T_1}{\partial y^2} + \frac{1}{r}\frac{\partial T_1}{\partial y}\right) \tag{14}$$

$$u_1 = v_1 = w_1 = T_1 = 0 \quad \text{at } y = 0 \tag{15a}$$

$$u_1 = T_1 = 0, \quad w_1 = U_\infty \sin 2\pi z/\lambda \quad \text{as } y \to \infty \tag{15b}$$

It should be noted that the model assumes a transverse disturbance imposed on a uniform outer flow with zero longitudinal pressure gradient. The axial pressure perturbation gradient is not needed‡ as the disturbance has been defined a priori. The crossflow pressure gradient and temperature gradient are $0(\varepsilon^2)$ and therefore do not appear in the first-order momentum and energy balances.

Using Eq. (8). a particular solution of Eq. (13) which satisfies the boundary conditions is seen to be

$$w_1 = u_0 \sin(2\pi z/\lambda) \tag{16}$$

The solution of the zero-order set of equations is well known.[22–25] The first-order equations contain the effect of the disturbance. This is evident in the products of zero- and first-order terms and the appearance of the crossflow velocity in the continuity equation. For the sake of completeness and continuity in coordinate systems a solution to both the zero- and the first-order sets of equations will be presented.

The equations are cast into an almost two-dimensional form by the use of a modified Mangler transformation[26] of the independent variables defined as

$$d\bar{x} = (r_0^2/L^2)\, dx \tag{17a}$$

$$d\bar{y} = (r/L)\, dy \tag{17b}$$

$$d\bar{\phi} = (r_0^2/L)(1/r_0)\, d\phi \tag{17c}$$

and

$$d\bar{z} = r_0\, d\bar{\phi} \tag{17d}$$

The transformed equations become

Zero order:

$$\partial u_0/\partial \bar{x} + \partial \bar{v}_0/\partial \bar{y} = 0 \tag{18}$$

$$u_0 \frac{\partial u_0}{\partial \bar{x}} + \bar{v}_0 \frac{\partial u_0}{\partial \bar{y}} = v\left\{\frac{\partial^2 u_0}{\partial \bar{y}^2} + \frac{2L}{r_0^2}\frac{\partial u_0}{\partial \bar{y}} + \frac{2L}{r_0^2}\bar{y}\frac{\partial^2 u_0}{\partial \bar{y}^2}\right\} \tag{19}$$

$$u_0 \frac{\partial \theta}{\partial \bar{x}} + \bar{v}_0 \frac{\partial \theta}{\partial \bar{y}} = \frac{v}{Pr}\left\{\frac{\partial^2 \theta}{\partial \bar{y}^2} + \frac{2L}{r_0^2}\frac{\partial \theta}{\partial \bar{y}} + \frac{2L}{r_0^2}\bar{y}\frac{\partial^2 \theta}{\partial \bar{y}^2}\right\} \tag{20}$$

where

$$\bar{v}_0 = (Lr/r_0^2) v_0 \quad \text{and} \quad \theta = (T_0 - T_w)/(T_\infty - T_w) \tag{21}$$

First order:

$$\partial u_1/\partial \bar{x} + \partial \bar{v}_1/\partial \bar{y} + \partial \bar{w}_1/\partial \bar{z} = 0 \tag{22}$$

‡ Tobak[16] has estimated the induced axial pressure gradient on a wedge due to a displacement surface which varies periodically in the transverse direction. For large x, the pressure coefficient is proportional to $x^{-1/2}$, however, it can be shown that for small x the pressure coefficient is independent of the axial coordinate.

$$u_0 \frac{\partial u_1}{\partial \bar{x}} + u_1 \frac{\partial u_0}{\partial \bar{x}} + \bar{v}_0 \frac{\partial u_1}{\partial \bar{y}} + \bar{v}_1 \frac{\partial u_0}{\partial \bar{y}} = v\left\{\frac{\partial^2 u_1}{\partial \bar{y}^2} + \frac{2L}{r_0^2}\frac{\partial u_1}{\partial \bar{y}} + \frac{2L}{r_0^2}\bar{y}\frac{\partial^2 u_1}{\partial \bar{y}^2}\right\} \tag{23}$$

$$u_0 \frac{\partial \Lambda}{\partial \bar{x}} + u_1 \frac{\partial \theta}{\partial \bar{x}} + \bar{v}_0 \frac{\partial \Lambda}{\partial \bar{y}} + \bar{v}_1 \frac{\partial \theta}{\partial \bar{y}} = \frac{v}{Pr}\left\{\frac{\partial^2 \Lambda}{\partial \bar{y}^2} + \frac{2L}{r_0^2}\frac{\partial \Lambda}{\partial \bar{y}} + \frac{2L}{r_0^2}\bar{y}\frac{\partial^2 \Lambda}{\partial \bar{y}^2}\right\} \tag{24}$$

where

$$\bar{v}_1 = (Lr/r_0^2) v_1, \quad \bar{w}_1 = (L/r) w_1, \quad \text{and} \quad \Lambda = T_1/(T_\infty - T_w) \tag{25}$$

The boundary conditions are

$$u_0 = \bar{v}_0 = u_1 = \bar{v}_1 = \bar{w}_1 = \theta = \Lambda = 0 \quad \text{at } \bar{y} = 0 \tag{26a}$$

$$u_0 = U_\infty, \quad \theta = 1, \quad \Lambda = 0, \quad u_1 = 0, \quad \bar{w}_1 = \frac{L}{r} U_\infty \sin\frac{2\pi z}{\lambda} \quad \text{as } \bar{y} \to \infty \tag{26b}$$

At this point, the partial differential equations are reduced to sets of ordinary differential equations by transforming the independent variables $\bar{x}$, $\bar{y}$ to ξ and η defined as

$$\xi = (2L/r_0^2)(v\bar{x}/U_\infty)^{1/2} \tag{27a}$$

$$\eta = \bar{y}(U_\infty/v\bar{x})^{1/2} \tag{27b}$$

In view of the initial supposition that the flowfield is considered to be an axisymmetric boundary-layer flow with transverse curvature and an imposed disturbance flow, a streamfunction is defined as

$$\psi = \psi_0 + \varepsilon\psi_1 + \ldots \tag{28}$$

where ψ_0 satisfies Eq. (18) and ψ_1 satisfies Eq. (22) with velocity components

$$u_1 = \partial\psi_1/\partial\bar{y} \tag{29a}$$

$$\bar{v}_1 = -\frac{\partial\psi_1}{\partial\bar{x}} - \int_0^{\bar{y}} \frac{\partial\bar{w}_1}{\partial\bar{z}} d\bar{y} \tag{29b}$$

In addition the streamfunctions are expanded§ as

$$\psi_0(\xi,\eta) = (vU_\infty \bar{x})^{1/2}[f_0(\eta) + \xi f_1(\eta) + \xi^2 f_2(\eta) + \ldots] \tag{30}$$

$$\psi_1(\xi,\eta) = (vU_\infty \bar{x})^{1/2}[g_0(\eta) + \xi g_1(\eta) + \xi^2 g_2(\eta) + \ldots] \tag{31}$$

and similarly the dimensionless temperatures are expanded as

$$\theta(\xi,\eta) = \theta_0(\eta) + \xi\theta_1(\eta) + \xi^2\theta_2(\eta) + \ldots \tag{32}$$

$$\Lambda(\xi,\eta) = \Lambda_0(\eta) + \xi\Lambda_1(\eta) + \xi^2\Lambda_2(\eta) + \ldots \tag{33}$$

The integral appearing in Eq. (29b) may also be expanded in a power series in ξ as

$$\Omega(\xi,\eta) = \Omega_0 + \xi\Omega_1 + \xi^2\Omega_2 + \ldots \tag{34}$$

$$\Omega = \left(\frac{v\bar{x}}{U_\infty}\right)^{1/2} \frac{2\pi}{\lambda}\frac{L^2}{r_0^2} U_\infty \cos\frac{2\pi z}{\lambda} \int_0^\eta \frac{f'(\eta)}{r} d\eta \tag{35}$$

Performing the transformation Eq. (27), using the expansions Eqs. (30–34) and equating to zero coefficients of the same power yields:

Zero order:

$$\xi^0 \begin{cases} f_0''' + \tfrac{1}{2} f_0 f_0'' = 0 & (36) \\ \theta_0'' + \tfrac{1}{2} Pr f_0 \theta_0' = 0 & (37) \end{cases}$$

$$\xi^1 \begin{cases} f_1''' + f_1 f_0'' + \tfrac{1}{2} f_0 f_1'' + \eta f_0''' + f_0'' - \tfrac{1}{2} f_1' f_0' = 0 & (38) \\ \theta_1'' + \tfrac{1}{2} Pr f_0 \theta_1' + Pr f_1 \theta_0' + \eta\theta_0'' + \theta_0' - \tfrac{1}{2} Pr f_0' \theta_1 = 0 & (39) \end{cases}$$

$$f_j'(0) = f_j(0) = \theta_j(0) = 0 \quad \text{at } \eta = 0 \quad \text{for } j \geq 0 \tag{40a}$$

$$f_0'(\infty) = \theta_0(\infty) = 1, \quad f_j' = \theta_j = 0 \quad \text{as } \eta \to \infty \quad j \geq 1 \tag{40b}$$

First order:

$$\xi^0 \begin{cases} g_0''' + \tfrac{1}{2} g_0 f_0'' + \tfrac{1}{2} g_0'' f_0 = -(\bar{x}/vU_\infty)^{1/2} \Omega_0 f_0'' & (41) \\ \Lambda_0'' + \tfrac{1}{2} Pr(f_0 \Lambda_0' + g_0 \theta_0') = -Pr(\bar{x}/vU_\infty)^{1/2} \Omega_0 \theta_0' & (42) \end{cases}$$

§ The same subscripting is used for the expansions in ε and ξ. In both cases, "0" is used to represent the leading term of the expansion.

$$g_0(0) = g_0'(0) = \Lambda_0(0) = f_0(0) = f_0'(0) = \theta_0(0) = 0 \quad \text{at } \eta = 0 \tag{43a}$$

$$g_0'(\infty) = \Lambda_0(\infty) = 0, \quad f_0'(\infty) = \theta_0(\infty) = 1.0 \quad \text{as } \eta \to \infty \tag{43b}$$

The ξ^1 set of equations has not been written here as it will contribute to the final solution only in the order of $(\varepsilon\xi)$.

As mentioned previously, Eqs. (36–39) with boundary conditions (40) have been previously solved and the solution is well known. Equations (36) and (37) are the equations for the flat plate boundary-layer flow, and Eqs. (38), (39) include the contribution of the transverse curvature. Equations (41) and (42) contain the effect of the imposed disturbance. It will be shown that in certain cases the contribution of the disturbance term can be significant relative to the transverse curvature contribution and indeed be a dominant effect if the crossflow velocity and Reynolds numbers are large.

An analytic solution to Eqs. (41) and (42) can be found for $Pr = 1$ provided the integral appearing in Eq. (35) can be evaluated. Expanding $1/r$ about $r = r_0$ (the cylinder radius) yields

$$\frac{1}{r} = \frac{1}{r_0}\left[1 - \left(\frac{r-r_0}{r_0}\right) + \left(\frac{r-r_0}{r_0}\right)^2 - \left(\frac{r-r_0}{r_0}\right)^3 + - \ldots\right] \tag{44}$$

where the range of interest is $r_0 \leq r \leq \delta$. Assuming the boundary layer to be thin, i.e., $\delta/r_0 \ll 1$, then

$$1/r \cong 1/r_0$$

and

$$\Omega = (\nu\bar{x}/U_\infty)^{1/2}(L^2/r_0^2)(2\pi/\lambda)U_\infty \cos(2\pi z/\lambda) f(\eta) \tag{45}$$

A solution¶ of Eq. (41) is therefore

$$g_0 = (2\pi x/\lambda)\cos(2\pi z/\lambda)(\eta f_0' - f_0) \tag{46}$$

Also, since $Pr = 1.0$ it can be shown that

$$\Lambda_0 = (2\pi x/\lambda)\cos(2\pi z/\lambda)\eta\theta_0' \tag{47}$$

The skin friction and Nusselt number (for $Pr = 1.0$) can now be written as

$$c_f = \frac{2}{(Re_x)^{1/2}}\left[f_0'' + \xi f_1'' + 0(\xi^2) + \frac{2\pi x}{\lambda}\varepsilon \cos\frac{2\pi z}{\lambda} f_0'' + 0(\varepsilon\xi)\right] \tag{48}$$

and

$$Nu_x = (Re_x)^{1/2}[\theta_0' + \xi\theta_1' + 0(\xi^2) + (2\pi/\lambda)(\nu/W_\infty)\varepsilon^2 \cos(2\pi z/\lambda) Re_x \theta_0' + 0(\varepsilon\xi)] \tag{49}$$

It is, of course, realized that if the skin friction and Nusselt number were integrated around the cylinder circumference the net contribution of the disturbance terms would be zero. What is of interest however, is the maximum and minimum values that occur locally around the cylinder (Fig. 2). Accordingly, if the cosine is integrated to obtain the average maximum value, the skin friction and Nusselt number can be put in the following form:

$$c_f = \frac{2}{(Re_x)^{1/2}}\left[f_0'' + \frac{2\nu}{r_0 U_\infty}(Re_x)^{1/2} f_1'' + \frac{4\varepsilon^2}{\lambda}\frac{\nu}{W_\infty} Re_x f_0''\right] \tag{50}$$

$$Nu_x = (Re_x)^{1/2}\left[\theta_0' + \frac{2\nu}{r_0 U_\infty}(Re_x)^{1/2}\theta_1' + \frac{4\varepsilon^2}{\lambda}\frac{\nu}{W_\infty} Re_x \theta_0'\right] \tag{51}$$

The result is applicable for thin boundary layers with $Pr = 1.0$. The ratio of the transverse curvature term to the disturbance term is

$$\frac{(c_f)\text{ transverse curvature}}{(c_f)\text{ disturbance}} = \frac{(Nu_x)\text{ transverse curvature}}{(Nu_x)\text{ disturbance}} = \frac{\lambda/\varepsilon}{2r_0(Re_x)^{1/2}}\frac{\theta_1'}{\theta_0'} \tag{52}$$

It is seen that the ratio is proportional to the curvature and disturbance wavelength and is inversely proportional to the square root of the Reynolds number. As would be expected, the relative influence of the disturbance term increases with increasing Reynolds number.

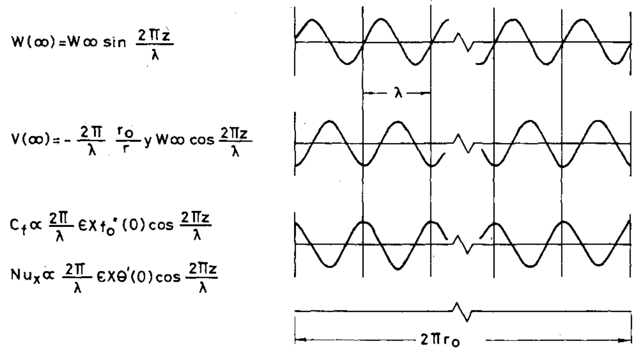

Fig. 2 Normal and crossflow velocity variation, skin friction and heat transfer with cylinder circumference.

¶ Solutions (46) and (47) have been previously given by Fannelop.[19] However, in Ref. 19, the term $(-f_0)$, Eq. (46), is missing.

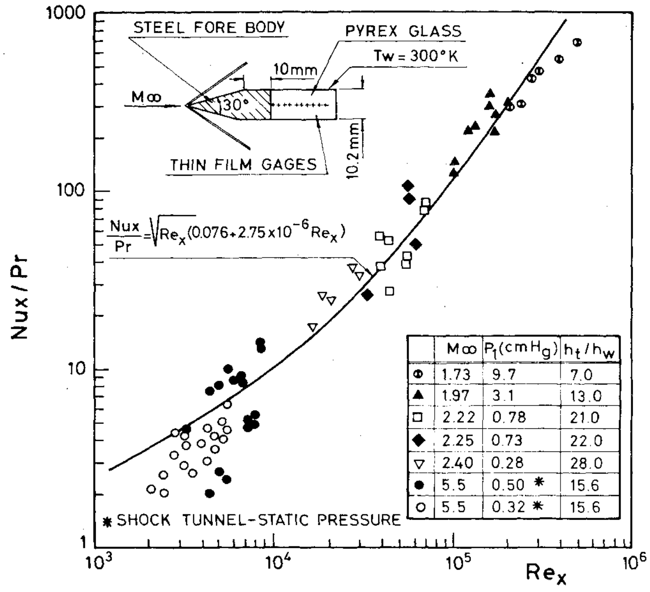

Fig. 3 Correlation of experimental data from Ref. 18.

III. Correlation of Experimental Results

In order to indicate the plausibility of the proposed model, the present theory is used to correlate heat-transfer data[18] obtained at the Technion shock tube/tunnel facility.[27] It should be noted that for the correlation of compressible data, Eq. (51) can only be used to indicate the functional variation of Nusselt number with Reynolds number. The coefficient θ_0' remains undetermined since the theory assumes an incompressible constant property flow. The Nusselt number vs Reynolds number variation for attached flow on a 30° cone-cylinder (radius = 5.1 mm) is reproduced in Fig. 3. The thin film platinum resistance thermometers used to determine the heat-transfer rates were located between 2.1 and 5.6 cylinder radii downstream of the expansion corner. The lower Reynolds number data corresponds to data obtained in the shock tunnel with a flow Mach number of 5.5 and freestream Reynolds numbers of 1.95×10^3/cm and 3.05×10^3/cm. The remainder of the data was obtained in the shock tube for flow Mach numbers 1.7 to 2.5, Reynolds numbers of 11×10^3/cm to 2×10^5/cm, and stagnation to wall enthalpy ratios of 7 to 30. The Nusselt and Reynolds numbers are evaluated at freestream conditions and the reference coordinate, x, is measured from the expansion corner.

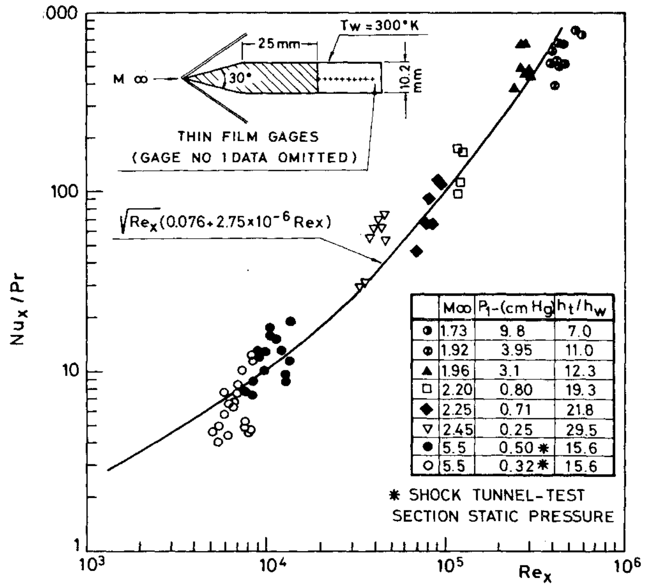

Fig. 4 Correlation of experimental data from Ref. 18.

Under the flow conditions present in the shock tube, a disturbed laminar boundary layer is suspected to occur. Streamline curvature necessary to enhance the disturbance is present at the overexpansion at the cone-cylinder corner. Furthermore, the Nusselt number exhibits an unusual Reynolds number dependance. Accordingly, it was felt that the proposed analytic model should correlate the experimental data.

Since neither the perturbation parameter ε nor the disturbance wavelength λ could be determined a priori, it was necessary to determine

$$c_1 = \theta_0'/Pr \tag{53a}$$

and

$$c_2 = (4\varepsilon^2/\lambda)(v/W_\infty)\theta_0'/Pr \tag{53b}$$

empirically from the experimental data. The constants were found to be 0.076 and 2.75×10^{-6} for c_1 and c_2, respectively, and therefore the Nusselt number is given by

$$Nu_x/Pr = (Re_x)^{1/2}(0.076 + 2.75 \times 10^{-6} Re_x) \tag{54}$$

The lower heat-transfer rates ($c_1 = 0.076$) are attributed to the thickening of the boundary layer due to the corner expansion and downstream interaction processes.[28–31] It should be noted that this is essentially a phenomenon of compressible boundary layers. In the incompressible case, the principal effect of a favorable pressure gradient is an increase in velocity and hence a thinning of the boundary layer. For the compressible, hypersonic boundary layer turning through small angles, the principal effect is a decrease in density and therefore an increase in displacement thickness. The present low values of heat-transfer rate are consistent with those predicted by Zakkay[28,29] and Sullivan.[30] In both cases heat-transfer rates after the corner expansion are approximately one-third the pre-expansion or flat plate values.

Equation (54) also fits the data obtained on a similar model with the platinum film gages located between 5.05 and 8.45 cylinder radii downstream of the corner. This is shown in Fig. 4. Since the level of values is the same for the two series of data, it is concluded that the assumption of negligible pressure gradient in the axial direction is reasonable.

It is instructive to examine the characteristics of the solution. For low Reynolds numbers the Nusselt number is essentially equal to $0.076(Re_x)^{1/2}$. From Eq. (54) it can be shown that there is an inflection point in the curve at Reynolds number equal to 9.2×10^3, which means that the solution rapidly attains the flat plate slope for the low Reynolds number range. This is consistent with physical reasoning. It should be noted that the solution in the lower Reynolds number range is somewhat of an approximation, since for the present analysis, the transverse curvature effect has been neglected. At higher Reynolds numbers, the disturbance term is the dominant factor.

Of interest at this point is an order of magnitude estimation of the wavelength λ. Table 1 shows the estimated values of the wavelength and wavenumber with the corresponding values of freestream Mach number, Nusselt number, stagnation to wall enthalpy ratio and unit Reynolds number. The wall curvature is the same for all points ($r_0 = 5.1$ mm). Equations (53) were used to calculate λ. The constants c_1 and c_2 were determined from the data and ε was assumed for this estimation to be $0(10^{-2})$. The wavenumber was calculated using Eqs. (53) in the following form:

$$2\pi r_0/\lambda = (\pi/2)(r_0/\varepsilon)(c_2/c_1)\,U_\infty/v \tag{55}$$

The wavenumber variation was therefore uniquely determined by the unit Reynolds number.

Heat-transfer data[31] obtained for reattaching flows has shown the same Reynolds number dependence as for the present attached flow on a cone-cylinder. The assumption of $\delta/r_0 \ll 1$ is a severe restriction for this particular case and no comparison is made at this time. The significance of crossflow velocities in reattachment geometries is currently being investigated and will be the topic of further research.

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Uniform Representation of the Gravitational Potential and its Derivatives

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The expressions for the gravitational potential and its derivatives of a nonspherical body in terms of spherical coordinates prove cumbersome at or near the zonal poles. A coordinate system is presented herein which removes this difficulty yet retains the numerical values and functional forms of the mass coefficients ($C_{n,m}$, $S_{n,m}$) as well as the orthogonality and recursion properties similar to the spherical harmonic representation.

Introduction

THE coordinates are used to express the potential and its first and second partial derivatives with respect to the Cartesian coordinates of a point mass. Recursion equations are derived for the new functions replacing the associated Legendre polynomials of the declination, or colatitude, and the trigonometric functions of the longitude. Moreover, the recursion equations are so chosen as to be stable for large values of the integer argument. Reference 1 contains a review of several methods for generating the first and second partials of a nonspherical body potential in a recursive formulation. The main advantage of the method outlined herein lies in a simplification of the formulation, which results in requiring only thirteen summation functions in place of the fourteen outlined in Ref. 1.

In order to facilitate the application of the new representation to statistical estimation theory for the determination of the Cartesian state and the mass coefficients, the necessary matrices and the variational differential equations are derived for a massless orbiter about a nonspherical, rotating body.

The attention of the author has been brought to a paper by Vinti,[2] in which the singularity of polar orbits is removed by a transformation in the complex domain and applied to the closed form solution of a satellite in orbit about an oblate gravitational body.

The Conventional Representation

Let the reference frame be an orthogonal Cartesian three-dimensional coordinate system fixed in the nonspherical body. In the most general case, the origin need not be at the center

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